

Statistical Mechanics of Interfering Links

M. B. Hastings

Center for Nonlinear Studies and Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545,
hastings@lanl.gov
(September 8, 2004)

We consider the statistical mechanics of interfering transmissions in a wireless communications protocol. In this case, a connection between two nodes requires all other nodes within communication distance of the given two nodes to remain quiet on the given channel. This leads to an interesting problem of dimers on a lattice, with a restriction that no two dimers can overlap or be nearest neighbors. We consider both an equilibrium and a non-equilibrium, “greedy” dynamics for the links; the equilibrium properties of the model are found to exhibit an interesting spin glass transition at maximum density on certain lattices, while the greedy construction is related to the problem of random sequential adsorption.

The study of networks has identified a variety of interesting networks with different topological properties [1], with recent interest considering dynamical processes such as flows on the networks [2]. Wireless communication networks provide a practically important example of information flow on a network. In this paper, we consider the problem of the MAC (media access control) layer, which lies below the routing layer and allows different nodes access to given frequencies. This layer must enable a large number of nodes to communicate over a finite number of frequencies with finite signal range, while optimizing the latency, throughput, and fairness of the protocol [3]. In this paper, we show that even the simplest model of the MAC layer gives rise to rich behavior, including glassy and non-equilibrium dynamics.

Recently, the problem of the MAC layer in such a protocol has been cast as a matching problem [4]. Consider some set of nodes, each with given signal range. As a simple approximation, the signal range leads to a directed network of possible communication links between nodes: there is a link from node i to node j if node j can receive a transmission from node i . As a further simplification, we consider the case in which the network is *undirected*, so that whenever node j can receive node i , then node i can also receive node j .

Suppose there is only one allowed frequency. In this case, suppose two nodes, i and j establish a communication link. Then, all nodes which are within range of node i or node j must remain silent, staying off the frequency in question, to avoid interference. Thus, it is possible for a given subset of links to be active simultaneously if and only if, for all nodes i , if node i is in an active link then i has exactly one neighbor j which is also active. Equivalently, this is the problem of finding a set of dimers in which nodes in different dimers are at least a distance of 2 from each other on the network; the presence of a dimer indicates an active link between the two nodes. The problem of multiple frequencies is similar, and leads to a coloring problem [5]. In Fig. 1, we show an example network and various possibilities of both allowed and disallowed sets of communication links.

In this paper, we study the statistical mechanics of this problem. We consider two different dynamics to generate the links. In the first case, we consider summing over all possible allowed sets of dimers, weighting the covering by a fugacity for the number of dimers. By taking the fugacity to be large, we are thus able to find high density configurations of dimers on the network. We solve this problem when the network is a Bethe lattice. Interestingly, the results depend on the coordination number of the Bethe lattice: for coordination number greater than 5, there appears to be a spin glass transition at a finite fugacity, while for coordination number less than 5, this transition is *fixed* at infinite fugacity. We also consider the problem on various finite dimensional lattices and speculate on the phase diagram. The problem is related to a recently considered statistical mechanics of dimers in the limit of infinite repulsion between neighboring dimers [6], but we identify the novel possibility of the spin glass phase in this limit.

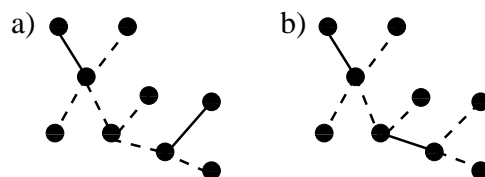


FIG. 1. a) An example of an allowed configuration of dimers. Solid lines denote the presence of a dimer (active link), dashed lines denote a link on the network which has no dimer (an inactive link). b) A disallowed configuration of dimers. The two dimers are within a distance one of each other.

The second dynamics we consider is a “greedy” dynamics, in which links are added one at a time to the network, each link being added at random among the allowed possibilities. This dynamics is a version of random sequential adsorption [7]. We will see that the greedy dynamics does very poorly on graphs with high coordination number.

To motivate the equilibrium and greedy dynamics, we start with the following more general set of dynamics:

at every instant, we suppose that some set of dimers is present. We then suppose that dimers can be created, when a link is activated, and disappear, when two nodes which were communicating stop using the channel. Both processes happen with given rates as follows. We suppose that at every instant of time each link which is inactive has some rate of activating (if that does not interfere with other links), and we set this rate to unity. We assign each dimer some rate of disappearing in any instant of time, and we set this rate to $1/\phi$. Then, in the stationary distribution of this dynamics, the probability of finding any given allowed configuration of dimers is $Z^{-1}\phi^{N_D}$, where N_D is the number of dimers in the given configuration and

$$Z = \sum_c \phi^{N_D(c)}, \quad (1)$$

where c is a configuration of dimers and $N_D(c)$ is the number of dimers in the configuration.

Then, the larger ϕ is, the larger the average number of active links will be, thus increasing throughput. However, a very large ϕ means that the configuration of active links changes slowly in time; the system will tend to remain stuck in a given configuration for a long time, which is also undesirable. We consider two cases below. First, the stationary dynamics at finite ϕ . Next, we consider $\phi = \infty$, and start from an initial configuration with no links active; in this case, links can be activated but are never deactivated. This leads to a random sequential adsorption problem, and we will see that the final density of links in this case is far below the maximum possible in finite ϕ .

Equilibrium Dynamics on the Bethe Lattice— We start with the equilibrium case, solving for the partition function (1). This problem is equivalent to that studied in [6] in the limit $\alpha \rightarrow -\infty$ (in that paper's notation). However, we consider the possibility of different spin glass and ordering transitions in this paper.

We consider a Bethe lattice with fixed branching ratio, k , so that every node has $q = k + 1$ neighbors. There are a number of different possible boundary conditions considered in the literature [8]. One possibility is to assume that the lattice indeed is a tree with given branching ratio and with some number of leaves; in this case, appropriate boundary conditions must be assigned at the leaves of the tree and one must consider only spins which are remote from the boundary. This procedure is reasonable in unfrustrated systems, and in general for systems outside of a spin glass phase. However, frustrated systems in the spin glass phase are sensitive to the boundary conditions which fix the degree of frustration [9]. Then, a better choice may be a random graph with fixed connectivity $k + 1$.

We will solve the problem with a set of recurrence equations on the Bethe lattice, and identify the onset of a spin glass phase. In the spin glass phase, which we do not

treat, the system becomes sensitive to boundary conditions. Outside the spin glass phase, the simple solution with recurrence equations is adequate to describe both the case of a tree with random boundary conditions and the case of a random graph.

To obtain the recurrence equations, we consider a tree with one of three possible states for the root of the tree (the root has only k neighbors). First, the root can be connected to one of its daughters via a dimer (we call this active). Second, the root can be not connected to any of its daughters, but have at least one of its daughters involved in an active link (we refer to this as locked). Third, the root can be not connected to any of its daughters, and have none of its daughters active either (we refer to this as free). These three possibilities are shown in Fig. 2(a-c).

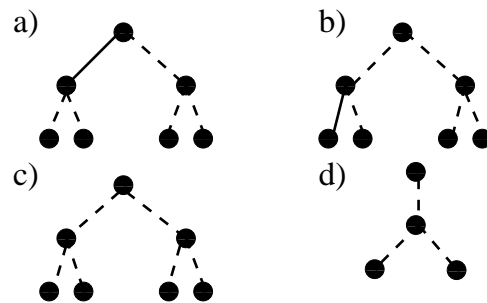


FIG. 2. a) Active configuration: root connected to daughter by an active link. b) Locked configuration: daughter involved in an active link, preventing root from forming a link to any other node. c) Free configuration: none of the daughters have active links. d) Joining $k + 1$ trees as described in text.

Let the partition function of the dimer problem on a tree in each of the three cases be Z_1, Z_2, Z_3 respectively. Then take k of these trees and join them together to make a tree of one higher level, and compute the resulting partition functions $\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3$ for the larger tree with the three different boundary conditions for that tree.

We have that $\tilde{Z}_1 = \phi k Z_3 (Z_2 + Z_3)^{k-1}$, as one of the k daughters must be free to have a dimer with the new root, and all the other daughters must be locked or free (they cannot be active or the root could not make a connection to the free daughter). Then, $\tilde{Z}_3 = (Z_2 + Z_3)^k$, as each of the k daughters of the new root must be either locked or free, but cannot be active, so each subtree has partition function $Z_2 + Z_3$. Finally, $\tilde{Z}_2 = (Z_1 + Z_2 + Z_3)^k - (Z_2 + Z_3)^k$, as in this case at least one daughter must be active; so, we consider all possible states of the daughters, $(Z_1 + Z_2 + Z_3)^k$, and subtract off those states with no daughters active, $\tilde{Z}_2 = (Z_2 + Z_3)^k$.

It is convenient to rescale the partition functions Z_i by a constant in looking for a fixed point. We choose to scale $\tilde{Z}_i \rightarrow \tilde{Z}_i / \tilde{Z}_3$, so $\tilde{Z}_3 = 1$. The rescaled equations become:

$$\begin{aligned}\tilde{Z}_3 &= 1; \quad \tilde{Z}_1 = \phi k(Z_2 + 1)^{k-1}/(Z_2 + 1)^k; \\ \tilde{Z}_2 &= (Z_1 + Z_2 + 1)^k - (Z_2 + 1)^k/(Z_2 + 1)^k.\end{aligned}\quad (2)$$

We have numerically solved Eqs. (2) and found that for small ϕ the solution converges to a fixed point, which we refer to as a liquid phase. For larger ϕ , there is a range for which the fixed point is stable, but there is also a stable set of solutions which *oscillate* with period three. The oscillating solution corresponds to a solid phase with broken translational symmetry, in which at one given level of the tree most of the nodes are free, at the next higher level most of the nodes are active to a daughter, and at the next higher level most of the nodes are locked.

At even higher values of ϕ , the stationary solution becomes unstable and only the solid solution remains. The existence of the solid solutions depends on the boundary conditions on the tree. If the lattice actually is a balanced tree with given branching ratio and depth, then the system has a phase transition to a solid phase. However, on a system on a random graph, the presence of loops will frustrate the solid phase. Alternately, on a tree in which the boundary conditions on the leaves are chosen sufficiently randomly (for example, the tree is not balanced, but instead unbalanced, so that different leaves are at different depths below the root), again we can stabilize the liquid phase. For given ϕ , there is only one liquid phase (the possibility of a liquid-gas transition as in [6] does not happen in this system).

At large ϕ , we can solve Eq. (2) in the liquid phase giving $Z_1^{liquid} = k^{(k+1)/(2k+1)}\phi^{(k+1)/(2k+1)}$, $Z_2^{liquid} = k^{k/(2k+1)}\phi^{k/(2k+1)}$, $Z_3^{liquid} = 1$. To find the probability that a given site has a dimer leaving it, we must then connect $q = k + 1$ of these trees into 1 tree, as in Fig. 2(d), rather than k trees as above. After carrying this out, one finds that for large ϕ a site has probability $(k + 1)/(2k + 1) - \mathcal{O}(\phi^{-1/(2k+1)})$ of having an active link emanating from it. Defining N to be the total number of sites and $\rho = 2N_D/N$, we get a limiting dimer density at large ϕ of $\rho = (k + 1)/(2k + 1) = q/(2q - 1)$.

The limiting density in the liquid phase can be understood on physical grounds. For *any* lattice with fixed coordination number $k + 1$ and with no loops of length 3, the dimer density is at most $(k + 1)/(2k + 1)$: consider a pair of sites 1, 2 connected by a dimer. Each of those two sites connects to k other sites, giving a total of $2k$ other sites connecting to the given two sites 1 and 2 (by assuming the absence of loops of length 3, the sites neighboring site 1 are distinct from those neighboring site 2). Thus, for each dimer, we can identify $2k$ sites which neighbor the given dimer; the total number of sites which neighbor dimers is then at least $2k/(k + 1)$: since each site has $k + 1$ neighbors, we may have overcounted the number of sites which neighbor dimers by a factor $k + 1$. Then, the total number of sites which have active links, plus the total number of sites which neighbor a site with an active link is at least $N_D[2 + 2k/(k + 1)] \leq N$. Thus,

$\rho = 2N_D/N \leq (k + 1)/(2k + 1)$. To achieve this density, it is necessary that if a node does not have an active link then all $k + 1$ of its neighbors have active links as in Fig. 3(a). While it appears that the system does achieve this maximum density as $\phi \rightarrow \infty$ from the results above, we now consider the possibility of a spin glass transition in the system which may prevent it from reaching the maximum density.

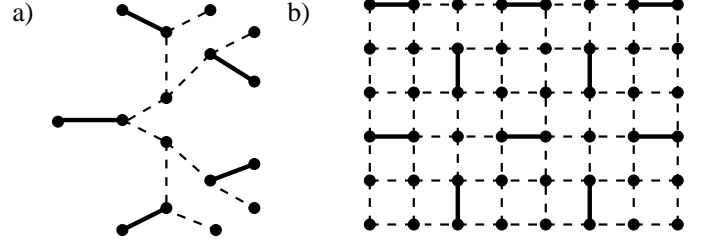


FIG. 3. a) Configuration to obtain highest density on Bethe lattice. b) Highest possible density configuration on the square lattice.

To identify the spin glass transition, we follow the procedure used for the Ising spin glass [10,8]. Let each of the k trees that we join together in the procedure above have slight random variations in the values of Z_1, Z_2, Z_3 about the liquid solution above, resulting from randomness in the boundary conditions. If the liquid solution is stable against these random fluctuations, then the Bethe lattice and random graph will behave identically in the thermodynamic limit. Otherwise, we go to a spin glass phase, and the complete graph may have different behavior. Fixing $Z_3 = 1$ as above, we have two physical quantities, Z_1 and Z_2 for each subtree. Suppose first that on each subtree each of these two values has *the same* slight fluctuation about the liquid solution above. We can compute the change in \tilde{Z}_1 and \tilde{Z}_2 to linear order from Eq. (2). The resulting linear transformation is found to have eigenvalues

$$\lambda^{\pm} = -\frac{k \pm \sqrt{k^2 - 4k}}{2} \quad (3)$$

at $\phi = \infty$. Let these eigenvalues have left eigenvectors v^{\pm} . Following [10], we consider the mean-square fluctuations $M^{\pm} \equiv |\sum_i v_i^{\pm} (Z_i - Z_i^{liquid})|^2$. Here, the overline denotes averaging over the random fluctuations, while the sum over $i = 1, 2$ projects $(Z_i - Z_i^{liquid})$ onto the corresponding right eigenvector. If on each subtree the fluctuations about the mean are random, then the mean-square fluctuation at the next higher level is given by $\bar{M}^{\pm} = |\lambda|^2/k M^{\pm}$, where the reduction by k is caused by averaging k random numbers together. When $|\lambda|^2 = k$, we have a spin glass transition. Now, for $k < 4$, the two eigenvalues are complex conjugates with absolute value \sqrt{k} at $\phi = \infty$, so that the spin glass transition occurs precisely at $\phi = \infty$, with no transition at $\phi < \infty$.

For $k > 4$, we have a spin glass transition at finite ϕ . The eigenvalues are both real in this case, with only one of them greater than \sqrt{k} in absolute value; this eigenvalue is negative. The presence of only one such eigenvalue suggests that the phase transition to the spin glass phase might be in the same universality class as the Ising spin glass on the random graph.

Equilibrium in Finite Dimensions— The treatment above is on the Bethe lattice. The properties of the system on finite dimensional systems are also of interest. On a square lattice, the system has a number of possible ordered structures. One can show that the pattern of Fig. 3(b) is the highest possible density. Thus, we conjecture that as a function of ϕ the system has a phase transition from a liquid state at low ϕ to the ordered state of Fig. 3(b) at large ϕ , with possible additional transitions to other ordered phases. A very interesting question is whether the system may have a glassy phase at some intermediate ϕ .

Greedy Dynamics— The second problem we consider is the random sequential adsorption case at $\phi = \infty$. Our technique follows that of [11] in a slightly different case of no interaction between dimers. We start at time $t = 0$ with no links on the lattice. We then add allowed links at a unit rate per link. We consider the probability $E_m(t)$ that after time t a connected cluster of m sites has (1): no links on any of the m sites and (2): none of the neighbors of the given m sites has a link. These conditions imply that the m sites are free to participate in links as they are not blocked by any of their neighbors. It is interesting that the probability can be written as $E_m(t)$, independent of the structure of the connected cluster (there are many possible clusters with m sites). At $t = 0$, we have $E_m(t) = 1$ for all m . At later times, one can derive the following differential equation for $m > 0$:

$$\partial_t E_m = -(m-1)E_m - [(k-1)m+2]E_{m+1} - [(k-1)m+2]kE_{m+2}. \quad (4)$$

The first term corresponds to the possibility of adding a dimer to any of the $(m-1)$ bonds that connect two sites, both in the cluster. The second term corresponds to the possibility of connecting a dimer to any of the $[(k-1)m+2]$ bonds which connect a site in the cluster to a site outside the cluster, while the last term corresponds to the possibility that a site which neighbors the cluster will connect to one of the k sites outside the cluster which neighbor the given site.

The ansatz $E_m(t) = c(t)\sigma(t)^{m-1}$ solves Eq. (4) with $\partial_t \sigma = -\sigma - (k-1)\sigma^2 - (k-1)k\sigma^3$ and $\partial_t c = -(k+1)\sigma c - (k+1)k\sigma^2 c$ and initial conditions $c(0) = \sigma(0) = 1$.

Define $\rho(t) = 2N_D(t)/N$ to be the fraction of sites with a dimer emanating from them. We find $\partial_t \rho = (k+1)E_2$.

Numerical solution of the differential equation gives $\rho(t \rightarrow \infty) \approx .313$ for $k = 4$. This is noticeably worse than the limiting value of $\rho = 5/9$ computed in the equilibrium

case above. At larger k , the non-equilibrium dynamics does even worse. The differential equations can be solved asymptotically at large k , where for $c, \sigma \gg 1/k$ the equations simplify to $\partial_t \sigma = -k^2 \sigma^3$, $\partial_t c = -k^2 \sigma^2 c$. The result for large k is

$$\rho(t \rightarrow \infty) \sim \ln(k)/k. \quad (5)$$

Discussion— We have studied a simple model inspired by recent developments in wireless communication. This model turns out to be a model of interacting dimers very similar to other models studied in statistical physics. From the point of view of statistical physics, one of the most interesting features of the model is the spin glass transition at $\phi = \infty$ for $k \leq 4$. In the language of statistical physics, this implies that the transition happens at zero temperature in the mean-field limit. This may simplify the treatment of this transition and may be useful for studying finite dimensional spin glasses.

The spin glass transition is of interest for practical purposes also. In the liquid phase heuristic algorithms will likely do a good job of rapidly finding allowed dimer configurations of given density, but in the glass phase it will be much more difficult to find good configurations and the system will tend to get stuck in certain configurations. Finally, the behavior of this system on regular lattices and random geometric graphs [12] in finite dimensions may be interesting.

Acknowledgements— I thank E. Ben-Naim, M. Marathe and V. S. Anil Kumar for useful discussions. This work was supported by DOE grant W-7405-ENG-36.

-
- [1] R. Albert and A. L. Barabasi, *Rev. Mod. Phys.* **74**, 47 (2002).
 - [2] Z. Toroczkai and K. E. Bassler, *Nature* **428**, 716 (2004).
 - [3] See P. Gupta and P. R. Kumar, *IEEE Trans. Inf. Thy.* **46**, 388 (2000) and references therein.
 - [4] H. Balakrishnan, C. Barrett, V. S. Anil Kumar, M. Marathe, and S. Thite, *IEEE Journal on Selected Areas in Communication*, to be published.
 - [5] C. Barrett, G. Istrate, V.S. Anil Kumar, M. Marathe and S. Thite, LANL Technical Report LA-UR-03-7654.
 - [6] A. B. Harris and M. Cohen, preprint cond-mat/0408618.
 - [7] J. W. Evans, *Rev. Mod. Phys.* **65**, 1281 (1993).
 - [8] M. Mezard and G. Parisi, *Eur. Phys. J. B.* **20**, 217 (2001).
 - [9] P. Y. Lai and Y. Y. Goldschmidt, *J. Phys. A* **22** (1989), 399.
 - [10] D. J. Thouless, *Phys. Rev. Lett.* **56**, 1082 (1986).
 - [11] S. N. Majumdar and V. Privman, *J. Phys. A* **26**, L743 (1993).
 - [12] Matthew D. Penrose, *Random Geometric Graphs*, (Oxford University Press, Oxford, 2003).